

Conformal Ricci and Matter Collineations for Anisotropic Fluid

M. Sharif *and Naghmana Tehseen

Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract

We study the consequences of timelike and spacelike conformal Ricci and conformal matter collineations for anisotropic fluid in the context of General Relativity. Necessary and sufficient conditions are derived for a spacetime with anisotropic fluid to admit conformal Ricci and conformal matter collineations parallel to u^a and x^a . These conditions for timelike and spacelike conformal Ricci and conformal matter collineations for anisotropic fluid reduce to the conditions of perfect fluid when the heat flux and the traceless anisotropic stress tensor vanish. Further, for $\alpha = 0$ (the conformal factor), we recover the earlier results of Ricci collineations and matter collineations in each case of timelike and spacelike conformal Ricci collineations and conformal matter collineations for the perfect fluid. Thus our results give the generalization of the results already available in the literature. It is worth noticing that the conditions of conformal matter collineations can be derived from the conditions of conformal Ricci collineations or vice versa under certain constraints.

Keywords: Conformal collineations, Anisotropic fluid.

PACS: 04.20.Jb Exact solutions.

*msharif@math.pu.edu.pk

1 Introduction

In General Relativity (GR), symmetries are used to understand the natural relationship between geometry and matter given by the Einstein field equations (EFEs). Symmetric system has not only the advantage of certain simplicity or even beauty but also special physical effects frequently occur. The symmetries are important in the classification of spacetime. Symmetries of geometrical/physical quantities are known as collineations. A symmetry (collineation) is defined by a relation

$$\mathcal{L}_\xi \phi = A,$$

where \mathcal{L} is the Lie derivative operator, ξ is the symmetry or collineation vector, ϕ is any of the quantities g_{ab} , R^a_{bcd} , R_{ab} , Γ^a_{bc} , T_{ab} and geometric objects constructed from them, A is a tensor with the same symmetries as ϕ . When $\phi_{ab} = g_{ab}$ and $A_{ab} = 2\alpha g_{ab}$, the symmetry vector ξ is called conformal Killing vector (CKV) and specializes to KV when $\alpha = 0$. When $\phi_{ab} = R_{ab}$ and $A_{ab} = \alpha R_{ab}$, the symmetry vector ξ is called conformal Ricci collineation (CRC) or Ricci inheritance collineation and specializes to Ricci collineation (RC) for $\alpha = 0$.

$$\mathcal{L}_\xi R_{ab} = \alpha R_{ab}. \quad (1.1)$$

When $\phi_{ab} = T_{ab}$ and $A_{ab} = \alpha T_{ab}$, where T_{ab} is the energy momentum tensor, the vector ξ is called conformal matter collineation (CMC) or matter inheritance collineation and becomes matter collineation (MC) when $\alpha = 0$.

$$\mathcal{L}_\xi T_{ab} = \alpha T_{ab}. \quad (1.2)$$

The function α in the case of CKVs is called conformal factor and in the case of conformal or inheriting collineations the conformal or inheriting factor. It is mentioned here that we are using the term *conformal Ricci or conformal matter collineation* instead of *Ricci or matter inheritance collineation*.

The study of inheritance symmetries with CRC and CMC in fluid spacetimes has recently attracted some interest. Oliver and Davis, [1,2] gave necessary and sufficient conditions for a matter spacetime to admit an RC. Letelier [3] discussed anisotropic fluids with two perfect fluid component. Herrera and Ponce [4-6] studied CKVs with particular reference to perfect and anisotropic fluids with different combination of fluids. Maartens et al. [7] made a study of SCKVs in anisotropic fluid. Carot et al. [8] investigated spacetime with CKVs. Coley and Tupper [9] studied spacetimes admitting

SCKV and symmetry inheritance. Duggal [10,11] discussed curvature inheritance symmetry and timelike CRC in perfect fluid spacetime. The theory of spacelike congruence in GR was first formulated by Greenberg [12] who discovered its application to the vortex congruences in a rotational fluid. This was developed further and applications to spacelike CKVs in spacelike congruence were considered by Mason and Tsamparlis [13]. Yavuz and Yilmaz [14] and Yilma et al. [15] considered CKVs and SCKVs and worked on the conformal curvature symmetry in string cosmology. Yilmaz [16] also studied timelike and spacelike collineations in string cloud.

Tsamparlis and Mason [17] investigated RCs in fluid spacetimes (perfect, imperfect and anisotropic fluids). Tsamparlis [18] discussed the conditions on the kinematic quantities of the congruence of the vector field generating the collineation. Baysal and Yilmaz [19] worked on spacelike conformal Ricci collineation (SpCRC) in a model of string fluid and string stress tensor. The same authors [20] also studied timelike and SpRCs in the model of string fluid. Sariddakis and Tsamparlis [21] discussed the applications for SpCKV and matter described either by perfect fluid or by an anisotropic fluid. Recently, Tsamparlis [22] discussed general symmetries of string fluid spacetime. In a recent paper, Sharif and Umber [23] have investigated timelike and SpCMC in specific forms of the energy-momentum tensor. They discussed the necessary and sufficient conditions for a vector to be timelike and spacelike admitting CMC.

The main purpose of such kind of work is the simplification of the EFEs to find the exact solutions. We describe the geometrization of a symmetry in terms of necessary and sufficient conditions on the geometry of the integral lines of the vector field that generates the symmetry. This enables us to express symmetry in a form which is convenient to the simplification of the field equations in a direct and inherent way. The symmetry can be studied in most convenient way if we take the Lie derivative of the field equations with respect to the vector u^a or ξ^a that generates the symmetry. This yields an expression such that the left hand side contains the Lie derivative of the Ricci tensor and the right hand side has the Lie derivative of the energy-momentum tensor. Consequently, we have the field equations as Lie derivatives along the symmetry vector of the dynamical variables.

In this paper, we shall elaborate the necessary and sufficient conditions for the existence of timelike and spacelike CRC and CMC for anisotropic fluid. The layout of the paper is the following. In section 2, we review the $1+1+2$ decomposition and consider the decomposition of the quantities which will be

used in later sections. In section 3, we investigate the kinematic conditions for timelike and spacelike CRCs and CMCs. Section 4 is devoted to derive the necessary and sufficient conditions for anisotropic fluid spacetime which admit CRCs. In section 5, we derive these conditions for anisotropic fluid spacetime which admit CMCs. Finally, section 6 contains summary and discussion of the results.

2 Notation

Let us consider a spacelike four position vector x^a and the corresponding unit timelike four-velocity vector u^a . Both the vectors are perpendicular to each other satisfying the relations given as follows

$$u^a u_a = -1, \quad u^a x_a = 0, \quad x^a x_a = 1. \quad (2.1)$$

The kinematical quantities σ_{ab} , ω_{ab} , \dot{u}_a and θ are defined [24] as

$$\dot{u}_a = u_{a;n} u^n = \frac{Du_a}{D\tau}, \quad (2.2)$$

$$\omega_{ab} = u_{[a;b]} + \dot{u}_{[a} u_{b]}, \quad (2.3)$$

$$\sigma_{ab} = u_{(a;b)} + \dot{u}_{(a} u_{b)} - \frac{\theta}{3} h_{ab}, \quad (2.4)$$

$$\theta = u^a_{;a}. \quad (2.5)$$

Using these quantities, the covariant differentiation of the four-vector velocity can be written in 1 + 3 decomposition as follows

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{\theta}{3} h_{ab} - \dot{u}_a u_b. \quad (2.6)$$

The projection tensor, $h_{ab} = g_{ab} + u_a u_b$, has the following properties:

$$h^{cd} x_c = x^d, \quad h^{cd} u_c = 0, \quad h^d_c h^b_d = h^b_c, \quad h_{cd} = h_{dc}, \quad h^c_c = 3. \quad (2.7)$$

If we use $\xi^a = \xi u^a$, the conformal Ricci and matter symmetries, respectively, can be re-written as

$$\dot{R}_{ab} + 2u^c R_{c(a} \ln \xi_{b)} + 2R_{c(a} u^c_{b)} = \beta R_{ab}, \quad (2.8)$$

$$\dot{T}_{ab} + 2u^c T_{c(a} \ln \xi_{b)} + 2T_{c(a} u^c_{b)} = \beta T_{ab}, \quad (2.9)$$

where $\beta = \frac{\alpha}{\xi}$. We define the screen projection operator normal to both u^a and x^a as

$$H_{ab} = h_{ab} - x_a x_b$$

which obeys the following properties:

$$H_{ab}u^a = 0 = H_{ab}x^a, \quad H_{ab}h_c^b = H_{ac}, \quad H_a^a = 2.$$

The $1 + 1 + 2$ decomposition of $x_{a;b}$ can be given as follows [25]

$$x_{a;b} = A_{ab} + x_a^* x_b - \dot{x}_a u_b + u_a [x^t u_{t;b} + (x^f \dot{u}_f) u_b - (x^f u_f^*) x_b], \quad (2.10)$$

where $s^* = s_{...;a} x^a$ and $A_{ab} = H_a^c H_b^d x_{c;d}$. The decomposition of A_{ab} into its irreducible parts is

$$A_{ab} = S_{ab} + B_{ab} + \frac{1}{2} \varepsilon H_{ab}, \quad (2.11)$$

where $S_{ab} = S_{ba}$, $S_b^b = 0$ is the traceless part (shear tensor), $B_{ab} = -B_{ab}$ is the antisymmetric part (rotation tensor) and ε is the trace (expansion). These quantities are given as

$$S_{ab} = (H_a^c H_b^d - \frac{1}{2} H^{cd} H_{ab}) x_{(c;d)}, \quad B_{ab} = H_a^c H_b^d x_{[c;d]}, \quad \varepsilon = H^{cd} x_{c;d}. \quad (2.12)$$

We can write the square bracket term in Eq.(2.10) as

$$- N_b + 2w_{fb} x^f + H_b^f \dot{x}_f, \quad (2.13)$$

where N_a is given by

$$N_a = H_a^b (\dot{x}_b - u_b^*). \quad (2.14)$$

This is called Greenberg vector [25]. Using Eq.(2.13) in Eq.(2.10), it follows that

$$x_{a;b} = A_{ab} + x_a^* x_b - \dot{x}_a u_b + H_b^c \dot{x}_c u_a + (2w_{tb} x^t - N_b) u_a. \quad (2.15)$$

Also, we have

$$x^f u_{f;b} = 2x^f u_{[f;b]} + u_b^* = -2\omega_{bf} x^f - (x_f \dot{u}^f) u_b + u_b^*. \quad (2.16)$$

When $\xi^a = \xi x^a$, the conformal Ricci and conformal matter symmetries can be written as

$$R_{ab}^* + 2x^c R_{c(a} \ln \xi_{b)} + 2R_{c(a} x_{b)}^c = \beta R_{ab}, \quad (2.17)$$

$$T_{ab}^* + 2x^c T_{c(a} \ln \xi_{b)} + 2T_{c(a} x_{b)}^c = \beta T_{ab}. \quad (2.18)$$

where $\beta = \frac{\alpha}{\xi}$.

The energy-momentum tensor of anisotropic fluid is given by [24]

$$T_{ab} = \rho u_a u_b + p h_{ab} + 2q_{(a} u_{b)} + \pi_{ab}, \quad (2.19)$$

where ρ is the total energy density, p denotes the isotropic pressure, q_a is the heat flux vector and π_{ab} is the traceless anisotropic stress tensor. The quantities u_a , q_a and π_{ab} satisfy the following relations:

$$u^a q_a = 0, \quad \pi_{ab} u^a = 0, \quad \pi_a^a = 0.$$

Using the Einstein field equations

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = T_{ab},$$

where Λ is a cosmological constant, the Ricci tensor of anisotropic fluid can be written as

$$R_{ab} = (\rho + p) u_a u_b + \frac{1}{2} (\rho - p + 2\Lambda) g_{ab} + 2q_{(a} u_{b)} + \pi_{ab}. \quad (2.20)$$

3 Timelike and Spacelike Conformal Ricci Collineations

This section is devoted to prove the necessary and sufficient conditions for the existence of timelike and spacelike CRC in the model of anisotropic fluid. In addition, we shall give the conditions for the existence of timelike and spacelike CRC in the form of kinematical quantities.

Proposition 1:

Anisotropic fluid spacetime with Ricci tensor, given by Eq.(2.20), admits a timelike CRC $\xi^a = \xi u^a$ if and only if

$$\dot{\rho} + 3\dot{p} + 2(\rho + 3p - 2\Lambda)(\ln \xi)^\cdot = \beta(\rho + 3p - 2\Lambda), \quad (3.1)$$

$$\begin{aligned} & (\rho + 3p - 2\Lambda)[\dot{u}_a - (\ln \xi)_{,a} - (\ln \xi)^\cdot u_a] + 2\dot{q}_a + 2q_a(\ln \xi)^\cdot \\ & + 2\dot{q}_b u^b u_a + 2q^b \omega_{ab} + 2q^b \sigma_{ab} + \frac{2}{3} \theta q_a = 2\beta q_a, \end{aligned} \quad (3.2)$$

$$\dot{\rho} - \dot{p} + \frac{2}{3}(\rho - p + 2\Lambda)\theta + \frac{4}{3}\dot{u}_f q^f - \frac{4}{3}q^f(\ln \xi)_{,f} + \frac{4}{3}\pi^{bf}\sigma_{fb}$$

$$= \beta(\rho - p + 2\Lambda), \quad (3.3)$$

$$\begin{aligned} & (\rho - p + 2\Lambda)\sigma_{ab} + 2q_{(a}\dot{u}_{b)} - \frac{2}{3}q^f\dot{u}_f h_{ab} - 2q_{(a}(\ln \xi)_{,b)} - 2u_{(a}q_{b)}(\ln \xi) \\ & + \frac{2}{3}\theta\pi_{ab} + \frac{2}{3}h_{ab}q^f(\ln \xi)_{,f} + \dot{\pi}_{ab} + 2\dot{\pi}_{f(a}u_{b)}u^f + 2\pi_{f(a}\omega_{b)}^f + 2\pi_{f(a}\sigma_{b)}^f \\ & + 2\pi_{f(a}\sigma_{b)}^f - \frac{2}{3}h_{ab}\pi^{cd}\sigma_{cd} = \beta\pi_{ab}. \end{aligned} \quad (3.4)$$

Proof:

First we assume that timelike CRC exists in this spacetime and prove that the conditions given by Eqs.(3.1)-(3.4) are satisfied.

When we substitute the value of Ricci tensor for anisotropic fluid from Eq.(2.20) in Eq.(2.8), we have

$$\begin{aligned} & (\rho + p)\dot{u}_a u_b + 2(\rho + p)u_{(a}\dot{u}_{b)} + \frac{1}{2}(\rho - p)\dot{g}_{ab} + 2q_{(a}\dot{u}_{b)} + 2\dot{q}_{(a}u_{b)} \\ & + \dot{\pi}_{ab} - (\rho + 3p - 2\Lambda)u_{(a}(\ln \xi)_{,b)} - 2q_{(a}(\ln \xi)_{,b)} + (\rho - p + 2\Lambda)u_{(a;b)} \\ & + 2q_f u_{(a}u_{;b)}^f + 2\pi_{f(a}u_{;b)}^f = \beta[(\rho + p)u_a u_b + \frac{1}{2}(\rho - p + 2\Lambda)g_{ab} \\ & + 2q_{(a}u_{b)} + \pi_{ab}]. \end{aligned}$$

Using 1 + 3 decomposition of $u_{a;b}$, the above expression can be written as

$$\begin{aligned} & [\frac{1}{2}(\dot{\rho} + 3\dot{p}) + (\rho + 3p - 2\Lambda)(\ln \xi)^{\cdot}]u_a u_b + [(\rho + 3p - 2\Lambda)(\dot{u}_c - (\ln \xi)_{,c}) \\ & + 2(\ln \xi)^{\cdot}q_c + 2\dot{q}_c + 2q^f(\sigma_{fc} + \omega_{fc} + \frac{1}{3}\theta h_{fc})]u_{(a}h_{b)}^c + \{\frac{1}{2}(\rho - p)\dot{h}_{cd} \\ & + (\rho - p + 2\Lambda)(\sigma_{cd} + \frac{1}{3}\theta h_{cd}) + \dot{\pi}_{cd} + \frac{2}{3}\theta\pi_{cd} + 2q_{(c}[\dot{u}_{d)} - (\ln \xi)_{,d)}]\} \\ & + 2\pi_{(fc}[\omega_{d)}^f + \sigma_{d)}^f]\}h_a^c h_b^d = \beta[\frac{1}{2}(\rho + 3p - 2\Lambda)u_a u_b + \frac{1}{2}(\rho - p + 2\Lambda)h_{ab} \\ & + 2q_{(a}u_{b)} + \pi_{ab}]. \end{aligned} \quad (3.5)$$

Contracting Eq.(3.5) in turn with $u^a u^b$, $u^a h_c^b$, h^{ab} and $h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd}$, we obtain

$$\dot{\rho} + 3\dot{p} + 2(\rho + 3p - 2\Lambda)(\ln \xi)^{\cdot} = \beta(\rho + 3p - 2\Lambda), \quad (3.6)$$

$$\begin{aligned} & (\rho + 3p - 2\Lambda)h_c^b[\dot{u}_b - (\ln \xi)_{,b}] + 2\dot{q}_c + 2\dot{q}_f u^f u_c \\ & + 2(\ln \xi)^{\cdot}q_c + 2q^f\omega_{fc} + 2q^f\sigma_{fc} + \frac{2}{3}q_c\theta = \beta q_c, \end{aligned} \quad (3.7)$$

$$\begin{aligned}
3\dot{\rho} &= 3\dot{p} + 2(\rho - p + 2\Lambda)\theta + 4\dot{u}_f q^f - 4q^f(\ln \xi), f \\
&+ 4\pi^{bf}\sigma_{fb} = 3\beta(\rho - p + 2\Lambda),
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
(h_c^a h_d^b &- \frac{1}{3}h^{ab}h_{cd})[(\rho - p + 2\Lambda)\sigma_{ab} + 2q_{(a}\dot{u}_{b)} + \dot{\pi}_{ab} \\
&- 2q_{(a}(\ln \xi)_{,b)} + 2\pi_{f(a}\omega_{b)}^f + 2\pi_{f(a}\sigma_{b)}^f + \frac{2}{3}\theta\pi_{ab}] \\
&= \beta[h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd}]\pi_{ab}.
\end{aligned} \tag{3.9}$$

Since

$$\begin{aligned}
(h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})\sigma_{ab} &= \sigma_{cd}, \\
(h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})(q_a\dot{u}_b + q_b\dot{u}_a) &= 2q_{(c}\dot{u}_{d)} - \frac{2}{3}q^b\dot{u}_b h_{cd}, \\
(h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})\dot{\pi}_{ab} &= \dot{\pi}_{cd} + 2\dot{\pi}_{b(c}u_{d)}u^b, \\
(h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})(\pi_{fa}\omega_b^f + \pi_{fb}\omega_a^f) &= 2\pi_{f(c}\omega_{d)}^f, \\
(h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})(\pi_{fa}\sigma_b^f + \pi_{fb}\sigma_a^f) &= 2\pi_{f(c}\sigma_{d)}^f - \frac{2}{3}h_{cd}\pi^{ab}\sigma_{ab}, \\
(h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})(q_a(\ln \xi)_{,b} + q_b(\ln \xi)_{,a}) &= 2q_{(c}(\ln \xi)_{,d)} + 2u_{(c}q_{d)}(\ln \xi) \\
&- \frac{2}{3}h_{cd}q^b(\ln \xi)_{,b}, \\
(h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})\pi_{ab} &= \pi_{cd}.
\end{aligned}$$

Substituting these values in Eq.(3.9), it follows that

$$\begin{aligned}
(\rho - p + 2\Lambda)\sigma_{cd} + 2q_{(c}\dot{u}_{d)} - \frac{2}{3}q^b\dot{u}_b h_{cd} + \dot{\pi}_{cd} + 2\dot{\pi}_{a(c}u_{d)}u^a \\
- 2q_{(c}(\ln \xi)_{,d)} - 2u_{(c}q_{d)}(\ln \xi) + \frac{2}{3}h_{cd}q^b(\ln \xi)_{,b} + 2\pi_{f(c}\omega_{d)}^f \\
+ 2\pi_{f(c}\sigma_{d)}^f - \frac{2}{3}h_{cd}\pi^{ab}\sigma_{ab} + \frac{2}{3}\theta\pi_{cd} = \beta\pi_{cd}.
\end{aligned} \tag{3.10}$$

(i) Eq.(3.6) is the same as Eq.(3.1).

(ii) Eq.(3.2) is obtained by expanding h_c^b in Eq.(3.7).

(iii) Eq.(3.8) is the same as Eq.(3.3).

(iv) Eq.(3.10) gives the condition (3.4).

Now we shall show that if the conditions (3.1)-(3.4) are satisfied, then there exists a timelike CRC in anisotropic fluid spacetime. For this purpose, we consider the left hand side of Eq.(3.5), i.e.,

$$\begin{aligned} & \left[\frac{1}{2}(\dot{\rho} + 3\dot{p}) + (\rho + 3p - 2\Lambda)(\ln \xi) \right] u_a u_b + [(\rho + 3p - 2\Lambda)(\dot{u}_c - \\ & (\ln \xi)_{,c}) + 2(\ln \xi) \dot{q}_c + 2\dot{q}_c + 2q^f(\sigma_{fc} + \omega_{fc} + \frac{1}{3}\theta h_{fc})] u_{(a} h_{b)}^c + \\ & \frac{1}{2}(\rho - p) \dot{h}_{cd} + (\rho - p + 2\Lambda)(\sigma_{cd} + \frac{1}{3}\theta h_{cd}) + \dot{\pi}_{cd} + \frac{2}{3}\theta \pi_{cd} \\ & + 2q_{(c}[\dot{u}_{d)} - (\ln \xi)_{,d}] + 2\pi_{(fc}[\omega_{d)}^f + \sigma_{d)}^f] h_a^c h_b^d. \end{aligned}$$

Now we make use of Eqs.(3.1)-(3.3) in the above expression so that

$$\begin{aligned} & (\rho - p + 2\Lambda)\sigma_{ab} + 2q_{(a}\dot{u}_{b)} - \frac{2}{3}q^f\dot{u}_f h_{ab} + \dot{\pi}_{ab} + 2\dot{\pi}_{f(a}u_{b)}u^f \\ & - 2q_{(a}(\ln \xi)_{,b)} - 2u_{(a}q_{b)}(\ln \xi) + \frac{2}{3}h_{ab}q^f(\ln \xi)_{,f} + 2\pi_{f(a}\omega_{b)}^f \\ & + 2\pi_{f(a}\sigma_{b)}^f - \frac{2}{3}h_{ab}\pi^{cd}\sigma_{cd} + \frac{2}{3}\theta\pi_{ab} + \beta[(\rho + p)u_a u_b \\ & + \frac{1}{2}(\rho - p + 2\Lambda)g_{ab} + 2q_{(a}u_{b)}]. \end{aligned} \quad (3.11)$$

Using Eq.(3.4), we finally obtain

$$\beta \left[\frac{1}{2}(\rho + 3p - 2\Lambda)u_a u_b + \frac{1}{2}(\rho - p + 2\Lambda)h_{ab} + 2q_{(a}u_{b)} + \pi_{ab} \right].$$

This is equal to the right hand side of Eq.(3.5). Hence the conditions (3.1)-(3.4) are necessary and sufficient for anisotropic fluid spacetime to admit a timelike CRC.

Proposition 2:

Anisotropic fluid with Ricci tensor, given by Eq.(2.20), admits a SpCRC $\xi^a = \xi x^a$ if and only if

$$\begin{aligned} & (\rho + 3p)^* + 2(\rho + 3p - 2\Lambda)x_a \dot{u}^a - 4q_a x^a (\ln \xi) - 4q^a N_a \\ & + 4q_a x^a u_d^* x^d = \beta(\rho + 3p - 2\Lambda), \\ & (\rho - p + 2\Lambda)[x_a^* + (\ln \xi)_{,a} - (\ln \xi)^* x_a] + 2(\pi_{ab} x^b)^* - 2\pi_{cd}^* x^c x^d x_a \end{aligned} \quad (3.12)$$

$$\begin{aligned}
& + 2\pi_{cd}x^cx^d(\ln \xi)_{,a} - 4\pi_{cd}x^cx^d(\ln \xi)^* + 2\pi_{ab}x^b(\ln \xi)^* + 2x^bq_b(\ln \xi)^*u_a \\
& - 2\pi_{cd}x^{*c}x^dx_a - 2\pi_{cd}\dot{x}^cx^du_a + 2x_c\dot{u}^cq_dx^du_a + 2\pi^{cd}x_dA_{ac} + 2q^cx^d\omega_{ad}x_c \\
& = 2\beta[\pi_{ab}x^b - \pi_{cd}x^cx^dx_a + q_bx^bu_a], \tag{3.13}
\end{aligned}$$

$$\begin{aligned}
(\rho & - p + 2\Lambda)N_a + 2(\rho + 3p - 2\Lambda)\omega_{ab}x^b + 2q_a\dot{x}^bu_b - 2q_cx^c\dot{x}^du_dx_a \\
& - 2q_a^* + 2q_c^*x^cx_a - 2q_c^*u^cu_a + 2\pi_{ab}^*u^bu^b + 2\pi_{cd}u^{*c}x^dx_a - 2q_bx^b(\ln \xi)_{,a} \\
& - 2q_bx^b(\ln \xi)u_a - 2q_bx^b(\ln \xi)^*x_a + 2\pi_{ba}x^b(\ln \xi) - 2\pi_{cd}x^cx^d(\ln \xi)\dot{x}_a \\
& + 2\pi_{ab}\dot{x}^b - 2\pi_{cd}x^c\dot{x}^dx_a - 2q^dA_{ad} = \beta(-q_a + q_bx^bx_a), \tag{3.14}
\end{aligned}$$

$$\begin{aligned}
(\rho & - p + 2\Lambda)[2(\ln \xi)^* - \varepsilon] + 3\pi_{cd}^*x^cx^d + 6\pi_{ab}x^bx^a(\ln \xi)^* + 2\pi_{ab}x^{*b}x^a \\
& - 2\pi^{ab}x_a(\ln \xi)_{,b} + 2x^aq^b\omega_{ab} - 2\pi^{ab}A_{ab} = 3\beta\pi_{ab}x^ax^b, \tag{3.15}
\end{aligned}$$

$$\begin{aligned}
(\rho & - p + 2\Lambda)S_{ab} + 2\pi_{c(a}A_{b)}^c - 2\pi^{cd}x_dA_{c(a}x_{b)} + 2\dot{x}^c\pi_{c(a}u_{b)} - 2H_{ab}\pi^{cd}A_{cd} \\
& + \pi_{ab}^* + 2u^c\pi_{c(a}^*u_{b)} - 2x^c\pi_{c(a}^*x_{b)} + \pi_{cd}^*x^cx^dx_ax_b - 2\pi_{cd}^*x^cu^dx_{(a}u_{b)} \\
& + \frac{1}{2}H_{ab}\pi_{cd}^*x^cx^d - 4x^cq_{(a}\omega_{b)c} + 4x^cq_dx^dx_{(a}\omega_{b)c} + 2H_{ab}\omega_{cd}q^cx^d \\
& + 2x^c\pi_{c(a}(\ln \xi)_{,b)} - 2x^cx^d\pi_{cd}(\ln \xi)_{,(a}x_{b)} + 2(\ln \xi)\dot{\pi}_{c(a}u_{b)}x^c \\
& - 2\pi_{cd}x^cx^d(\ln \xi)\dot{x}_{(a}u_{b)} - 2x^c(\ln \xi)^*\pi_{c(a}x_{b)} + 2x^c\pi_{cd}x^d(\ln \xi)^*x_ax_b \\
& - H_{ab}[x^c\pi^{cd}(\ln \xi)_{,d} - \pi_{cd}x^cx^d(\ln \xi)^*] = \beta[\pi_{ab} - 2x^c\pi_{c(a}x_{b)} \\
& + \pi_{cd}x^cx^dx_ax_b + \frac{1}{2}H_{ab}\pi_{cd}x^cx^d]. \tag{3.16}
\end{aligned}$$

This can be proved on the same lines as Proposition 1.

4 Timelike and Spacelike Conformal Matter Collineations

In this section we shall state and prove the necessary and sufficient conditions for the existence of timelike and spacelike CMC in the model of anisotropic fluid.

Proposition 3:

Anisotropic fluid spacetime with the energy-momentum tensor, given by Eq.(2.19), admits a timelike CMC $\xi^a = \xi u^a$ if and only if the following conditions are satisfied.

$$\begin{aligned}
\dot{\rho} & + 2\rho(\ln \xi)\dot{} = \beta\rho, \tag{4.1} \\
\rho[\dot{u}_a & - ((\ln \xi)_{,a} - (\ln \xi)\dot{}u_a)] + \dot{q}_a + \dot{q}_f u^f u_a + (\ln \xi)\dot{}q_a
\end{aligned}$$

$$+ q^f \omega_{fa} + q^f \sigma_{fa} + \frac{1}{3} \theta q_a = \beta q_a, \quad (4.2)$$

$$\dot{p} + \frac{2}{3} p \theta + \frac{2}{3} q^a \dot{u}_a - \frac{2}{3} q^a (\ln \xi)_{,a} + \frac{2}{3} \pi^{ab} \sigma_{ba} = \beta p, \quad (4.3)$$

$$\begin{aligned} 2p\sigma_{ab} &+ 2q_{(a} \dot{u}_{b)} - \frac{2}{3} q^f \dot{u}_f h_{ab} - 2q_{(a} (\ln \xi)_{,b)} - 2u_{(a} q_{b)} (\ln \xi) + \\ \frac{2}{3} \theta \pi_{ab} &+ \frac{2}{3} q^f (\ln \xi)_{,f} h_{ab} + \dot{\pi}_{ab} + 2\dot{\pi}_{c(a} u_{b)} u^c + 2\pi_{f(a} \omega_{b)}^f \\ &+ 2\pi_{f(a} \sigma_{b)}^f - \frac{2}{3} \pi_f^c \sigma_c^f h_{ab} = \beta \pi_{ab}. \end{aligned} \quad (4.4)$$

Proof:

First we assume that timelike CMC exists in this spacetime and prove that the conditions given by Eq.(4.1)-(4.4) are satisfied.

When we substitute value of the energy-momentum tensor for anisotropic fluid from Eq.(2.19) in Eq.(2.9), we have

$$\begin{aligned} (\rho + p) u_a u_b + 2(\rho + p) u_{(a} \dot{u}_{b)} + \dot{p} g_{ab} + 2q_{(a} \dot{u}_{b)} + 2\dot{q}_{(a} u_{b)} + \dot{\pi}_{ab} \\ - 2\rho u_{(a} (\ln \xi)_{,b)} - 2q_{(a} (\ln \xi)_{,b)} + 2p u_{(a;b)} + 2q_f u_{(a} u_{;b)}^f + 2\pi_{f(a} u_{;b)}^f \\ = \beta [(\rho + p) u_a u_b + p g_{ab} + 2q_{(a} u_{b)} + \pi_{ab}]. \end{aligned}$$

This implies that

$$\begin{aligned} [\dot{\rho} + 2\rho (\ln \xi)] u_a u_b + [2\rho (\dot{u}_c - (\ln \xi)_{,c}) + 2\dot{q}_c + 2(\ln \xi) \dot{q}_c \\ + 2q^f (\sigma_{fc} + \omega_{fc} + \frac{1}{3} \theta h_{fc})] u_{(a} h_{b)}^c + \{\dot{p} h_{cd} + 2p(\sigma_{cd} + \frac{1}{3} h_{cd}) \\ + \dot{\pi}_{cd} + \frac{2}{3} \theta \pi_{cd} + 2q_{(c} [\dot{u}_{d)} - (\ln \xi)_{,d)} + 2\pi_{f(c} [w_{d)}^f + \sigma_{d)}^f]\} h_a^c h_b^d \\ = \beta [\rho u_a u_b + p h_{ab} + 2q_{(a} u_{b)} + \pi_{ab}]. \end{aligned} \quad (4.5)$$

Contracting Eq.(4.5) in turn with $u^a u^b$, $u^a h_c^b h^{ab}$ and $h_c^a h_d^b - \frac{1}{3} h^{ab} h_{cd}$, we obtain

$$\dot{\rho} + 2\rho (\ln \xi) = \beta \rho, \quad (4.6)$$

$$\begin{aligned} \rho h_c^b [\dot{u}_b - (\ln \xi)_{,b}] + \dot{q}_c + (\ln \xi) \dot{q}_c + \dot{q}_f u^f u_c + q_f \omega_c^f \\ + q_f \sigma_c^f + \frac{1}{3} \theta q_c = \beta q_c, \end{aligned} \quad (4.7)$$

$$\begin{aligned} 3\dot{p} + 2p\theta + 2\dot{u}_f q^f - 2q^f (\ln \xi)_{,f} + 2\pi^{bf} \sigma_{fb} = 3\beta p, \quad (4.8) \\ (h_c^a h_d^b - \frac{1}{3} h^{ab} h_{cd}) [2q_{(a} \dot{u}_{b)} + \dot{\pi}_{ab} - 2q_{(a} (\ln \xi)_{,b)} + 2p\sigma_{ab} \end{aligned}$$

$$+2\pi_{f(a)\omega_b^f} + 2\pi_{f(a)\sigma_b^f} + \frac{2}{3}\theta\pi_{ab}] = \beta(h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})\pi_{ab}. \quad (4.9)$$

Since

$$\begin{aligned} (h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})\sigma_{ab} &= \sigma_{cd}, \\ (h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})(q_a \dot{u}_b + q_b \dot{u}_a) &= 2q_{(c} \dot{u}_{d)} - \frac{2}{3}q^b \dot{u}_b h_{cd}, \\ (h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})\dot{\pi}_{ab} &= \dot{\pi}_{cd} + 2\dot{\pi}_{b(c} u_{d)} u^b, \\ (h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})(\pi_{fa}\omega_b^f + \pi_{fb}\omega_a^f) &= 2\pi_{f(c}\omega_{d)}^f, \\ (h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})(\pi_{fa}\sigma_b^f + \pi_{fb}\sigma_a^f) &= 2\pi_{f(c}\sigma_{d)}^f - \frac{2}{3}h_{cd}\pi^{ab}\sigma_{ab}, \\ (h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})(q_a(\ln \xi)_{,b} + q_b(\ln \xi)_{,a}) &= 2q_{(c}(\ln \xi)_{,d)} + 2u_{(c}q_{d)}(\ln \xi) \\ &\quad - \frac{2}{3}h_{cd}q^b(\ln \xi)_{,b}, \\ (h_c^a h_d^b - \frac{1}{3}h^{ab}h_{cd})\pi_{ab} &= \pi_{cd}. \end{aligned}$$

Thus Eq.(4.9) takes the following form

$$\begin{aligned} 2p\sigma_{ab} + 2q_{(a}\dot{u}_{b)} - \frac{2}{3}q^f \dot{u}_f h_{ab} - 2q_{(a}(\ln \xi)_{,b)} - 2u_{(a}q_{b)}(\ln \xi) + \frac{2}{3}\theta\pi_{ab} \\ + \frac{2}{3}q^f(\ln \xi)_{,f}h_{ab} + \dot{\pi}_{ab} + 2\dot{\pi}_{c(a}u_{b)}u^c + 2\pi_{f(a}\omega_{b)}^f + 2\pi_{f(a}\sigma_{b)}^f \\ - \frac{2}{3}\pi^{cd}\sigma_{cd}h_{ab} = \beta\pi_{ab}. \end{aligned} \quad (4.10)$$

(i) Eq.(4.6) is the same as Eq.(4.1).

(ii) Eq.(4.2) is obtained by expanding Eq.(4.7).

(iii) Dividing Eq.(4.8) by 3, we get the condition (4.3).

(iv) Condition (4.4) is the same as Eq.(4.10).

Conversely, if the conditions (4.1)-(4.4) are satisfied, then there must exist a timelike CMC in an anisotropic fluid spacetime. For this purpose, we

consider the left hand side of Eq.(4.5)

$$\begin{aligned}
[\dot{\rho} &+ 2\rho(\ln \xi)]u_a u_b + [2\rho(\dot{u}_c - (\ln \xi)_{,c}) + 2\dot{q}_c + 2q^f(\sigma_{fc} + \omega_{fc} \\
&+ \frac{1}{3}\theta h_{fc})]u_{(a}h_{b)}^c + \{\dot{p}h_{cd} + 2p(\sigma_{cd} + \frac{1}{3}h_{cd}) + \dot{\pi}_{cd} + \frac{2}{3}\theta\pi_{cd} \\
&+ 2q_{(c}[\dot{u}_{d)} - (\ln \xi)_{,d}] + 2\pi_{f(c}[w_{d)}^f + \sigma_{d)}^f]\}h_a^c h_b^d.
\end{aligned}$$

Substituting the values from Eqs.(4.1)-(4.3) in the above expression, we obtain

$$\begin{aligned}
2p\sigma_{ab} &+ 2q_{(a}\dot{u}_{b)} - \frac{2}{3}q^f\dot{u}_f h_{ab} - 2q_{(a}(\ln \xi)_{,b)} - 2u_{(a}q_{b)}(\ln \xi) \\
&+ \frac{2}{3}\theta\pi_{ab} + \frac{2}{3}q^f(\ln \xi)_{,f}h_{ab} + \dot{\pi}_{ab} + 2\dot{\pi}_{c(a}u_{b)}u^c + 2\pi_{f(a}\omega_{b)}^f \\
&+ 2\pi_{f(a}\sigma_{b)}^f - \frac{2}{3}\pi^{cd}\sigma_{cd}h_{ab} + \beta[(\rho + p)u_a u_b + \frac{1}{2}(\rho - p + 2\Lambda)g_{ab} \\
&+ 2q_{(a}u_{b)}].
\end{aligned} \tag{4.11}$$

If we make use of Eq.(4.4), it follows that

$$\beta(\rho u_a u_b + p h_{ab} + 2q_{(a}u_{b)} + \pi_{ab})$$

which is equal to the right hand side of Eq.(4.5). Hence the conditions (4.1)-(4.5) are necessary and sufficient for anisotropic fluid spacetime to admit a timelike CMC.

It is interesting to note that if we replace $\frac{1}{2}(\rho + 3p - 2\Lambda)$ by ρ and $\frac{1}{2}(\rho - p + 2\Lambda)$ by p in Eqs. (3.1) to (3.4) then we obtain Eqs.(4.1) to (4.4). This means that we can determine CMC from CRC.

Proposition 4:

Anisotropic fluid with the energy-momentum tensor, given by Eq.(2.19), admits a SpCMC $\xi^a = \xi x^a$ if and only if

$$\begin{aligned}
\rho^* &+ 2\rho(\dot{u}^c x_c) - 2(q^c x_c)(\ln \xi) - 2(q^c N_c) + 2q^c x_c u^{*d} x_d = \beta\rho, \tag{4.12} \\
p[x_a^* &+ (\ln \xi)_{,a} - (\ln \xi)^* x_a] + (\pi_{ab} x^b)^* - \pi_{cd}^* x^c x^d x_a + \pi_{cd} x^c x^d (\ln \xi)_{,a} \\
&- 2\pi_{cd} x^c x^d (\ln \xi)^* x_a + \pi_{ba} x^b (\ln \xi)^* + x^c q_c (\ln \xi)^* u_a - \pi_{cd} x^* x^c x^d x_a \\
&- \pi_{cd} \dot{x}^c x^d u_a + x_c \dot{u}^b q_d x^b u_a + \pi^{cd} x_d A_{ac} + q_c x^d \omega_{cd} u_a \\
&= \beta[\pi_{ab} x^b - \pi_{cd} x^c x^d x_a + q_b x^b u_a], \tag{4.13} \\
pN_a &+ 2\rho\omega_{ab} x^b + q_a \dot{x}^b u_b - q_b x^b \dot{x}^c u_c x_a - q_a^* + q_c^* x^c x_a + q_b^* u^b u_a
\end{aligned}$$

$$\begin{aligned}
& + \pi_{ab}^* u^b + \pi_{cd} u^{*c} x^d x_a - q_b x^b (\ln \xi)_{,a} - q_b x^b (\ln \xi) u_a - q_b x^b (\ln \xi)^* x_a \\
& + \pi_{ab} x^b (\ln \xi) - \pi_{cd} x^c x^d (\ln \xi) x_a + \pi_{ab} \dot{x}^b - \pi_{cd} x^c \dot{x}^d x_a - q^b A_{cb} \\
& = \beta(-q_a + q_b x^b x_a), \tag{4.14} \\
4p(\ln \xi)^* & - 2p\varepsilon + 3\pi_{ab}^* x^a x^b + 6\pi_{ab} x^b x^a (\ln \xi)^* + 2\pi_{ab} x^a x^{*b} \\
& - 2x_a \pi^{ba} (\ln \xi)_{,b} - 4x_a q^b \omega_{ab} - 2\pi^{ab} A_{ab} = 3\beta \pi_{ab} x^a x^b, \tag{4.15} \\
2pS_{ab} & + 2\pi_{c(a} A_{b)}^c - 2\pi^{cd} x_d A_{c(a} x_{b)} + 2\dot{x}^c \pi_{c(a} u_{b)} - 2H_{ab} \pi^{cd} A_{cd} \\
& + \pi_{ab}^* + 2u^c \pi_{c(a}^* u_{b)} - 2x^c \pi_{c(a}^* x_{b)} + \pi_{cd}^* x^c x^d x_a x_b - 2\pi_{cd}^* x^c u^d x_{(a} u_{b)} \\
& + \frac{1}{2} H_{ab} \pi_{cd}^* x^c x^d - 4x^c q_{(a} \omega_{b)c} + 4x^c q_d x^d x_{(a} \omega_{b)c} + 2H_{ab} \omega_{cd} q^c x^d \\
& + 2x^c \pi_{c(a} (\ln \xi)_{,b)} - 2x^c x^d \pi_{cd} (\ln \xi)_{,(a} x_{b)} + 2(\ln \xi) \pi_{c(a} u_{b)} x^c \\
& - 2\pi_{cd} x^c x^d (\ln \xi) x_{(a} u_{b)} - 2x^c (\ln \xi)^* \pi_{c(a} x_{b)} + 2x^c \pi_{cd} x^d (\ln \xi)^* x_a x_b \\
& - H_{ab} [x_c \pi^{cd} (\ln \xi)_{,d} - \pi_{cd} x^c x^d (\ln \xi)^*] = \beta[\pi_{ab} - 2x^c \pi_{c(a} x_{b)} \\
& + \pi_{cd} x^c x^d x_a x_b + \frac{1}{2} H_{ab} \pi_{cd} x^c x^d]. \tag{4.16}
\end{aligned}$$

The proof follows similarly as for Proposition 3.

5 Outlook

Physically, there is a close connection of inheriting CKVs with the relativistic thermodynamics of fluids since for a distribution of massless particles in equilibrium, the inverse temperature function is inheriting CKV. This paper deals with the fundamental question of determining when the symmetries of the geometry is inherited by all the source terms of a prescribed matter tensor of EFEs. We have investigated timelike and spacelike CRCs and CMCs of anisotropic fluid using a particular procedure.

We formulate conditions defining the CRCs and CMCs for anisotropic fluid. It is mentioned here that when we take $\pi_{ab} = 0 = q_a$ in anisotropic fluid, the conditions for timelike and spacelike CRC reduce to the conditions of perfect fluid [25]. Also, for $\pi_{ab} = 0$, $q_a = 0$, the conditions for timelike and spacelike CMC reduce to the conditions of perfect fluid [23]. Further, for $\alpha = 0$, we obtain the conditions of RCs and MCs in each case of spacelike and timelike CRCs and CMCs for the perfect fluid. This shows that our results provide the generalization of the results already available in the literature. It is worthwhile to note that if we replace $\frac{1}{2}(\rho + 3p - 2\Lambda)$ by ρ and $\frac{1}{2}(\rho - p + 2\Lambda)$ by

p , then we can determine CMC from CRC. We have obtained the conditions for the existence of CRCs and CMCs in the models of anisotropic fluid. These conditions can be used as restriction for the EFEs. Since the non-linearity of EFEs ceases to extract their exact solution, the restricted equations may give interesting solution in respective spacetimes. It can be shown that a string fluid is the simplest example of anisotropic fluid with vanishing heat flux. It would be interesting to extend this procedure to the combination of two perfect fluids. This work is in preparation [26] and will appear elsewhere.

References

- [1] Oliver, D.T., and Davis, W.R.: Ann. Inst. Henri Poincare **3**(1977)399.
- [2] Oliver, D.T., and Davis, W.R.: Gen. Rel. Grav. **8**(1979)905.
- [3] Letelier, P.S.: Phys. Rev. **D22**(1980)807.
- [4] Herrera, L. and Ponce de, L.J.: J. Math. Phys. **26**(1985)778.
- [5] Herrera, L. and Ponce de, L.J.: J. Math. Phys. **26**(1985)2018.
- [6] Herrera, L. and Ponce de, L.J.: J. Math. Phys. **26**(1985)2847.
- [7] Maartens, R., Mason, D.P. and Tsamparlis, M.: J. Math. Phys. **27**(1986)2987.
- [8] Carot, J., Coley, A.A., and Sintes, A.: Gen. Rel. Grav. **28**(1986)311.
- [9] Coley, A.A. and Tupper, B.O.J.: J. Math. Phys. **30**(1989)2616.
- [10] Duggal, K.L.: J. Math. Phys. **33**(1992)2989.
- [11] Duggal, K.L.: Acta Appl. Math. **31**(1993)225.
- [12] Greenberg, P.S.: J. Math. Anal. Appl. **30**(1970)128.
- [13] Mason, D.P. and Tsamparlis, M.: J. Math. Phys. **26**(1985)2881.
- [14] Yavuz, I. and Yilmaz, I.: Gen. Rel. Grav. **29**(1997)1295.

- [15] Yilmaz, I., Tarhan, I., Yavuz, I., Baysal, H. and Camci, U.: Int. J. Mod. Phys. **D8**(1999)659.
- [16] Yilmaz, I.: Int. J. Mod. Phys. **D10**(2001)681.
- [17] Tsamparlis, M. and Mason, D.P.: J. Math. Phys. **31**(1990)1707.
- [18] Tsamparlis, M.: J. Math. Phys. **33**(1992)1472.
- [19] Baysal, H. and Yilmaz, I.: Class Quantum Grav. **19**(2002)6435.
- [20] Baysal, H. and Yilmaz, I.: Turk J. Phys. **27**(2003)83.
- [21] Saridakis, E. and Tsamparlis, M.: J. Math. Phys. **32**(1991)1541.
- [22] Tsamparlis, M.: Gen. Rel. Grav. **38**(2006)279.
- [23] Sharif, M. and Sheikh, Umber: Int. J. Mod. Phys. **A21**(2006)3213.
- [24] Stephani, Hans: *General Relativity: An Introduction to the Theory of the Gravitational Field* (Cambridge University Press, 1990).
- [25] Duggal, K.L. and Sharma R.: *Symmetries of Spacetime and Riemann Manifold* (Kluwer Academic Publisher, 1999).
- [26] Sharif, M. and Tehseen, Naghmana: *Conformal Ricci and Matter Collineations for Two Perfect Fluids*, submitted for publication.